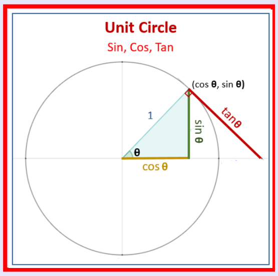
Math 241

Winter 2024

Lecture 17

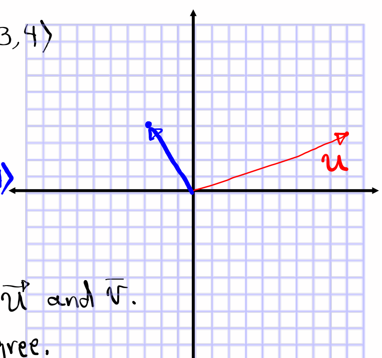


Feb 19-8:47 AM

Class QZ 12

Given $u = \langle 12, 5 \rangle$ and $v = \langle -3, 4 \rangle$

1) Draw u & v



2) Find $u + v = \langle 12 + {}^{-}3, 5 + 4 \rangle$

$= \langle 9, 9 \rangle$

3) Find the angle between $\vec{u}$ and $\vec{v}$.
Round to nearest whole degree.

$u \cdot v = 12(^{-}3) + 5 \cdot 4$

$= -36 + 20 = -16$

$\cos\theta = \dfrac{u \cdot v}{|u|\,|v|} = \dfrac{-16}{13 \cdot 5}$

$|u| = \sqrt{12^2 + 5^2} = 13$

$|v| = \sqrt{(-3)^2 + 4^2} = 5$

$\cos\theta = \dfrac{-16}{65} \rightarrow \theta = \cos^{-1}\left(\dfrac{-16}{65}\right)$

$\boxed{\theta \approx 104^\circ}$

Find $\text{Proj}_v u = \dfrac{u \cdot v}{|v|^2} v = \dfrac{-16}{5^2} \langle -3, 4 \rangle$

$= \dfrac{-16}{25} \langle -3, 4 \rangle = \left\langle \dfrac{48}{25}, \dfrac{-64}{25} \right\rangle$

Jan 31-6:59 AM

Given $Z = 4 - 3i$, $W = 2 + 5i$

1) $|Z| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = \boxed{5}$

2) $Z - W = 4 - 3i - 2 - 5i$
$= \boxed{2 - 8i}$

3) $ZW = (4-3i)(2+5i)$
$= 8 + 20i - 6i - 15i^2$
$= 8 + 14i - 15(-1)$
$= 8 + 14i + 15$
$= \boxed{23 + 14i}$

4) $\dfrac{Z}{W}$
$= \dfrac{4-3i}{2+5i} \cdot \dfrac{2-5i}{2-5i}$
$= \dfrac{8 - 20i - 6i + 15i^2}{4 - \cancel{10i} + \cancel{10i} - 25i^2}$
$= \dfrac{8 - 26i + 15(-1)}{4 - 25(-1)}$
$= \dfrac{-7 - 26i}{29}$
$= \boxed{\dfrac{-7}{29} - \dfrac{26}{29}i}$

Jan 31-8:23 AM

Simplify $i^{80} - i^{43}$

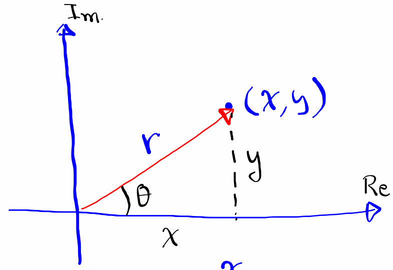
$= (i^2)^{40} - i^{42} \cdot i$
$= (-1)^{40} - (i^2)^{21} \cdot i$
$= 1 - (-1)^{21} \cdot i$
$= 1 - (-1)i = \boxed{1 + i}$

Jan 31-8:34 AM

$Z = x + yi$

x ↑ Real Part

yi ↑ Imaginary Part



$r = |Z| = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$\cos\theta = \dfrac{x}{r}$

$\sin\theta = \dfrac{y}{r}$

$\tan\theta = \dfrac{y}{x}$

$Z = x + yi = r\cos\theta + r\sin\theta\, i$

$= r(\cos\theta + i\sin\theta)$

Trig. Form of a Complex number

(Polar Form)

Jan 31-8:41 AM

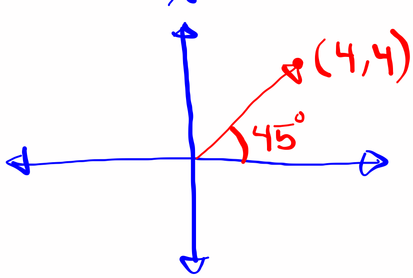
$Z = 4 + 4i$

$x = 4$

$y = 4$

$\rightarrow r = \sqrt{x^2+y^2} = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$

$\tan\theta = \dfrac{y}{x} \rightarrow \tan\theta = 1 \rightarrow \theta = 45^\circ$



$4+4i = r(\cos\theta + i\sin\theta)$

$= 4\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$

Jan 31-8:46 AM

$Z = -1 + i\sqrt{3}$

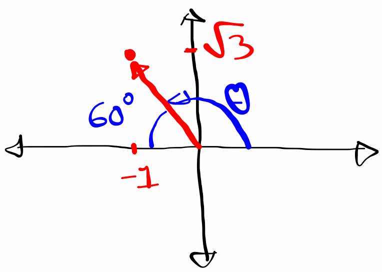
$x = -1$

$\rightarrow r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = \boxed{2}$

$y = \sqrt{3}$

$\tan\theta = \dfrac{y}{x}$ $\qquad \tan\theta = \dfrac{\sqrt{3}}{-1}$ $\qquad \tan\theta = -\sqrt{3}$

R.A. 60°



$\theta = 180^\circ - 60^\circ = 120^\circ$

$-1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$= \underbrace{2(\cos 120^\circ + i\sin 120^\circ)}_{\text{Polar Form}}$

Jan 31-8:49 AM

$Z = 4(\cos 300^\circ + i\sin 300^\circ)$ Polar Form

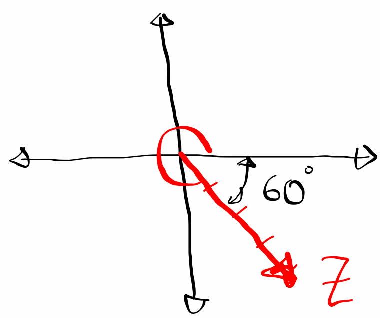
$r = 4$ $\qquad \theta = 300^\circ$

$\cos 300^\circ = \cos 60^\circ = \dfrac{1}{2}$

$\sin 300^\circ = -\sin 60^\circ = \dfrac{-\sqrt{3}}{2}$

$x = r\cos 300^\circ = 4 \cdot \dfrac{1}{2} = 2$

$y = r\sin 300^\circ = 4 \cdot \dfrac{-\sqrt{3}}{2} = -2\sqrt{3}$

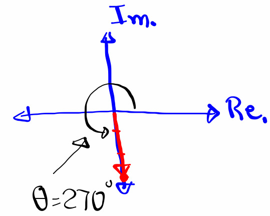


$\boxed{Z = 2 - 2i\sqrt{3}}$

Jan 31-8:55 AM

Given $Z = -3i = 0 - 3i$

1) Draw Z



2) $x = 0$

$y = -3$

$r = \sqrt{x^2 + y^2} = 3$

3) Find θ

$\tan\theta = \frac{y}{x} = \frac{-3}{0}$

undefined

$-3i = r(\cos\theta + i\sin\theta)$

$= 3(\cos 270^\circ + i\sin 270^\circ)$

Rectangular Form

$x + yi$

$r = \sqrt{x^2 + y^2}$

$x = r\cos\theta, \quad y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

Trig. Form

(Polar Form)

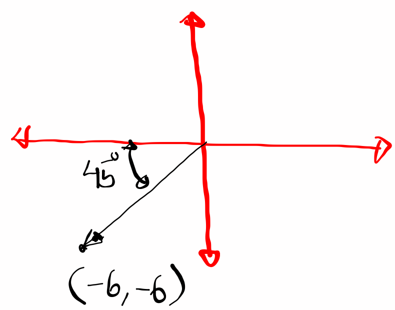
$r(\cos\theta + i\sin\theta)$

$\boxed{r\,cis\,\theta}$

Jan 31-8:59 AM

$Z = -6 - 6i$

1) Draw Z



2) $x = -6$

$y = -6$

$r = \sqrt{x^2 + y^2} = 6\sqrt{2}$

$-6 - 6i = r(\cos\theta + i\sin\theta)$

$= 6\sqrt{2}(\cos 225^\circ + i\sin 225^\circ)$

$\boxed{6\sqrt{2}\ cis\ 225^\circ}$

3) Find θ

$\tan\theta = \frac{y}{x}$ $\quad \tan\theta = \frac{-6}{-6}$

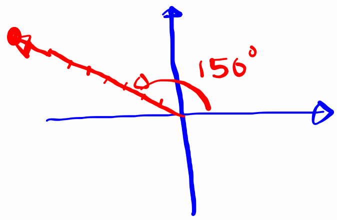
$\tan\theta = 1$ $\quad \theta = 180^\circ + 45^\circ$

R.A. 45° $\quad = 225^\circ$

Jan 31-9:10 AM

Given $Z = 8 \operatorname{cis} 150^\circ$

1) Draw Z



2) write Z in rectangular form.

$x + yi$

$$x = r\cos\theta = 8\cos 150^\circ = 8 \cdot -\cos 30^\circ = -8 \cdot \frac{\sqrt{3}}{2} = -4\sqrt{3}$$

$$y = r\sin\theta = 8\sin 150^\circ = 8 \cdot \sin 30^\circ = 8 \cdot \frac{1}{2} = 4$$

$$8 \operatorname{cis} 150^\circ = 8(\cos 150^\circ + i\sin 150^\circ)$$

$$= \boxed{-4\sqrt{3} + 4i}$$

Jan 31-9:16 AM

$$Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$Z_1 \cdot Z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

$$Z_1^n = r_1^n(\cos n\theta_1 + i\sin n\theta_1)$$

Jan 31-9:38 AM

$Z = 3(\cos 45^\circ + i \sin 45^\circ)$

$W = 2(\cos 30^\circ + i \sin 30^\circ)$

$ZW = 3 \cdot 2 (\cos \underbrace{75^\circ}_{45^\circ + 30^\circ} + i \sin 75^\circ)$

$= 6(\cos 75^\circ + i \sin 75^\circ)$

$\frac{Z}{W} = \frac{3}{2}(\cos \underbrace{15^\circ}_{45^\circ - 30^\circ} + i \sin 15^\circ)$

$Z^2 = 3^2(\cos \underbrace{2 \cdot 45^\circ}_{n\theta} + i \sin 2 \cdot 45^\circ)$

$= 9(\cos 90^\circ + i \sin 90^\circ)$

Jan 31-9:43 AM

$Z = 4 \text{ Cis } 100^\circ$

$W = 2 \text{ Cis } 50^\circ$

$ZW = 4 \cdot 2 \text{ Cis } (100^\circ + 50^\circ) = 8 \text{ Cis } 150^\circ$

$\frac{Z}{W} = \frac{4}{2} \text{ Cis } (100^\circ - 50^\circ) = 2 \text{ Cis } 50^\circ$

$W^3 = 2^3 \text{ Cis } 3 \cdot 50^\circ = 8 \text{ Cis } 150^\circ$

Simplify $\frac{3 \text{ Cis } 305^\circ}{9 \text{ Cis } 65^\circ} = \frac{1}{3} \text{ Cis } (305^\circ - 65^\circ)$

$= \frac{1}{3} \text{ Cis } 240^\circ$

Jan 31-9:48 AM

Simplify

$(12 \text{ Cis } 18.5°)(3 \text{ Cis } 11.5°)$

$= 12 \cdot 3 \text{ Cis} (18.5° + 11.5°)$

$= 36 \text{ Cis } 30° = 36[\cos 30° + i \sin 30°]$

$= 36\left[\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right]$

$= \boxed{18\sqrt{3} + 18i}$

Jan 31-9:54 AM

Simplify

$$\frac{45 \text{ Cis } \frac{3\pi}{4}}{22.5 \text{ Cis } \frac{\pi}{3}} = 2 \text{ Cis}\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)$$

$$= 2 \text{ Cis } \frac{5\pi}{12}$$

Jan 31-9:57 AM

$Z = 2\sqrt{6} - 2i\sqrt{2}$ Find $\frac{Z}{W}$

$W = \sqrt{2} - i\sqrt{6}$

Method I

$-2\sqrt{12}\,i^2$

$$\frac{2\sqrt{6} - 2i\sqrt{2}}{\sqrt{2} - i\sqrt{6}} \cdot \frac{\sqrt{2} + i\sqrt{6}}{\sqrt{2} + i\sqrt{6}} = \frac{2\sqrt{12} + 2i\sqrt{36} - 2i\sqrt{4}}{\sqrt{4} + \cancel{i\sqrt{12}} - \cancel{i\sqrt{12}} - i^2\sqrt{36}}$$

$$= \frac{4\sqrt{3} + 12i - 4i + 4\sqrt{3}}{2 + 6} = \frac{8\sqrt{3} + 8i}{8}$$

$$= \frac{8(\sqrt{3} + i)}{8}$$

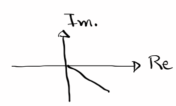
$$= \boxed{\sqrt{3} + i}$$

Jan 31-10:01 AM

$Z = 2\sqrt{6} - 2i\sqrt{2}$ Find $\frac{Z}{W}$

$W = \sqrt{2} - i\sqrt{6}$

Method II



$2\sqrt{6} - 2i\sqrt{2} = \boxed{4\sqrt{2}\ \text{Cis}\ 330°}$

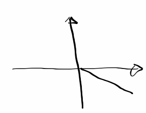
$x = 2\sqrt{6}$ $\quad r = \sqrt{24+8} = \sqrt{32} = 4\sqrt{2}$

$y = -2\sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{2}}{2\sqrt{6}} = -\frac{1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$ $\quad$ R.A. $30°$

$\theta = 360° - 30° = 330°$

$\sqrt{2} - i\sqrt{6} = 2\sqrt{2}\ \text{Cis}\ 300°$



$x = \sqrt{2}$ $\quad \rightarrow r = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2}$

$y = -\sqrt{6}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{6}}{\sqrt{2}} = -\sqrt{3}$ $\quad$ R.A. $60°$

$\theta = 360° - 60° = 300°$

$$\frac{Z}{W} = \frac{4\sqrt{2}\ \text{Cis}\ 330°}{2\sqrt{2}\ \text{Cis}\ 300°} = 2\ \text{Cis}(330° - 300°)$$

$$= 2\ \text{Cis}\ 30°$$

$$= 2[\cos 30° + i\sin 30°]$$

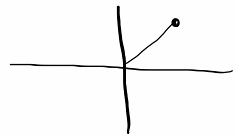
$$= 2\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right)$$

$$\boxed{\sqrt{3} + i}$$

Jan 31-10:09 AM

Simplify $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^8 = \left(1 \operatorname{Cis} 45^\circ\right)^8$

$\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$



$= 1^8 \operatorname{Cis} 8 \cdot 45^\circ$

$x = \frac{\sqrt{2}}{2}$

$= \operatorname{Cis} 360^\circ$

$\rightarrow r = \sqrt{\frac{2}{4} + \frac{2}{4}} = \sqrt{\frac{4}{4}} = 1$

$y = \frac{\sqrt{2}}{2}$

$= \operatorname{Cos} 360^\circ + i \operatorname{Sin} 360^\circ$

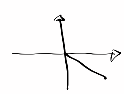
$\tan\theta = \frac{y}{x}$ $\quad \tan\theta = 1$ $\quad = 1 + i \cdot 0 = \boxed{1}$

$\theta = 45^\circ$

Jan 31-10:17 AM

$(\sqrt{3} - i)^5$

$\sqrt{3} - i$



$x = \sqrt{3}$

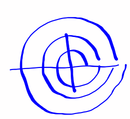
$\rightarrow r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

$y = -1$

$\tan\theta = \frac{y}{x}$ $\quad \tan\theta = \frac{-1}{\sqrt{3}}$ $\quad \tan\theta = -\frac{\sqrt{3}}{3}$ $\quad$ R.A. 30°

$\theta = 360^\circ - 30^\circ = 330^\circ$

$(\sqrt{3} - i)^5 = (2 \operatorname{cis} 330^\circ)^5$



$= 2^5 \operatorname{Cis} 5 \cdot 330^\circ$

$\frac{1650}{360} = 4.58\overline{3}$

$= 32 \operatorname{Cis} 1650^\circ$

4 times around, then 210°

$.58\overline{3}(360^\circ) = 210^\circ$

$= 32 \operatorname{Cis} 210^\circ$

$= 32[\operatorname{Cos} 210^\circ + i \operatorname{Sin} 210^\circ]$

$= 32[-\operatorname{Cos} 30^\circ - i \operatorname{Sin} 30^\circ]$

$= 32\left[-\frac{\sqrt{3}}{2} - i \cdot \frac{1}{2}\right]$

$= \boxed{-16\sqrt{3} - 16i}$

Jan 31-10:22 AM

nth root of $z = r \text{ Cis } \theta$

$\sqrt[n]{z} = \sqrt[n]{r} \text{ Cis } \alpha$

$\alpha = \dfrac{\theta + 360^\circ \cdot K}{n}$ $\quad K = 0, 1, 2, 3, \dots, n-1$

$\boxed{\text{De MoiVre Thrm}}$

De – moi – vreh

Jan 31-10:30 AM

$z^3 = 8$

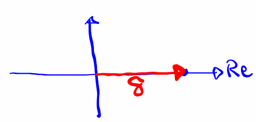
One answer is $\sqrt[3]{8} = 2$

There are two more answers.

$8 = 8 + 0i$

Re $x = 8$

Im. $y = 0$



$r = 8, \quad \theta = 0^\circ$

$8 = 8 \text{ Cis } 0^\circ$

$\alpha = \dfrac{\theta + 360^\circ \cdot K}{n}$

$= \dfrac{0 + 360^\circ \cdot K}{3}$

$= \dfrac{360^\circ K}{3} = 120^\circ K$

$\sqrt[3]{8} = \sqrt[3]{8} \text{ Cis } \alpha$

$= 2 \text{ Cis } \alpha$

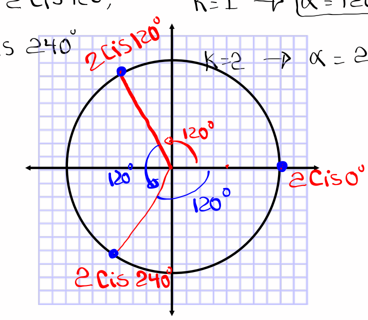
$K = 0 \rightarrow \boxed{\alpha = 0}$

$2 \text{ Cis } 0^\circ, \; 2 \text{ Cis } 120^\circ,$

$K = 1 \rightarrow \boxed{\alpha = 120^\circ}$

$2 \text{ Cis } 240^\circ$

$K = 2 \rightarrow \alpha = 240^\circ$



Jan 31-10:34 AM

Find all 4th roots of $16 \text{ Cis } 120^\circ$

$n=4 \rightarrow \sqrt[4]{}$ $\quad$ r $\quad$ θ

$\sqrt[4]{16 \text{ Cis } 120^\circ} = \sqrt[4]{16} \text{ Cis } \alpha$ $\quad \sqrt[4]{16} = 2$

$\alpha = \dfrac{\theta + 360^\circ \cdot K}{4}$ $\quad K = 0, 1, 2, 3$

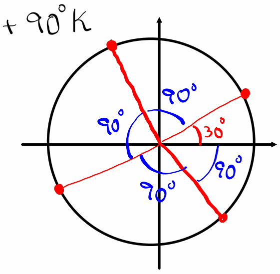
$\alpha = \dfrac{120^\circ + 360^\circ \cdot K}{4} = 30^\circ + 90^\circ K$

$K=0 \rightarrow \alpha = 30^\circ$

$K=1 \rightarrow \alpha = 120^\circ$

$K=2 \rightarrow \alpha = 210^\circ$

$K=3 \rightarrow \alpha = 300^\circ$



Final Ans. $\quad 2 \text{ Cis } 30^\circ$

$2 \text{ Cis } 120^\circ$

$2 \text{ Cis } 210^\circ$

$2 \text{ Cis } 300^\circ$

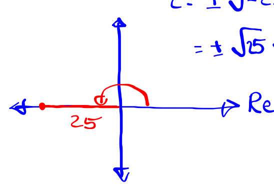
Jan 31-10:42 AM

Solve $\quad Z^2 + 25 = 0$

$Z = \pm\sqrt{-25}$

$= \pm\sqrt{25}\sqrt{-1}$

$= \boxed{\pm 5i}$



$Z^2 = -25$

$-25 = -25 + 0i$

$x = -25 \qquad r = 25$

$y = 0 \qquad \theta = 180^\circ$

$Z^2 = 25 \text{ Cis } 180^\circ$

$Z = \sqrt{25 \text{ Cis } 180^\circ} = 5 \text{ Cis } \alpha$

$n=2$

$\alpha = \dfrac{180^\circ + 360^\circ \cdot K}{2}$

$\alpha = 90^\circ + 180^\circ K$

$K=0 \rightarrow \alpha = 90^\circ \rightarrow \boxed{5 \text{ Cis } 90^\circ}$

$K=1 \rightarrow \alpha = 270^\circ \rightarrow \boxed{5 \text{ Cis } 270^\circ}$

$5 \text{ Cis } 90^\circ = 5(\overset{0}{\cancel{\cos 90^\circ}} + i\,\overset{1}{\cancel{\sin 90^\circ}}) = 5i$

$5 \text{ Cis } 270^\circ = 5[\overset{0}{\cancel{\cos 270^\circ}} + i\,\overset{-1}{\cancel{\sin 270^\circ}}] = -5i$

Jan 31-10:50 AM

Solve $z^2 = 2 + 2i$

$z^2 = 2 \text{ Cis } 45°$

$z = \sqrt{2 \text{ Cis } 45°} = \sqrt{2} \text{ Cis } \alpha$

$n = 2$

$\alpha = \dfrac{45° + 360° \cdot K}{2}$

$K = 0 \quad \alpha = 22.5°$

$\alpha = 22.5° + 180° \cdot K$

$K = 1 \quad \alpha = 202.5°$

$\sqrt{2} \text{ Cis } 22.5°, \ \sqrt{2} \text{ Cis } 202.5°$

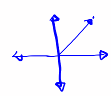
Jan 31-11:00 AM

Solve $x^4 - (8 + 8i\sqrt{3}) = 0$

4 answers

$x = 8$

$y = 8\sqrt{3}$



$x^4 = 8 + 8i\sqrt{3}$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{8^2 + 8^2 \cdot 3}$

$x^4 = 16 \text{ Cis } 60°$

$n = 4$

$= \sqrt{256} = 16$

$x = \sqrt[4]{16 \text{ Cis } 60°}$

$\theta = RA = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{8\sqrt{3}}{8}$

$= \tan^{-1} \sqrt{3} = 60°$

$= 2 \text{ Cis } \alpha \qquad \alpha = \dfrac{60° + 360° \cdot K}{4}$

$\alpha = 15° + 90° K$

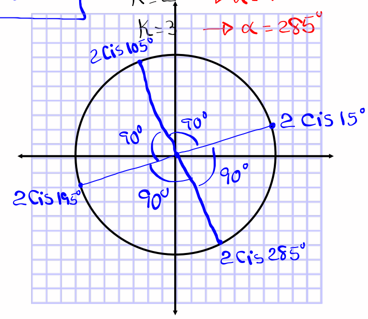
$\boxed{\begin{array}{l} 2 \text{ Cis } 15° \\ 2 \text{ Cis } 105° \\ 2 \text{ Cis } 195° \\ 2 \text{ Cis } 285° \end{array}}$

$K = 0 \rightarrow \alpha = 15°$

$K = 1 \rightarrow \alpha = 105°$

$K = 2 \rightarrow \alpha = 195°$

$K = 3 \rightarrow \alpha = 285°$

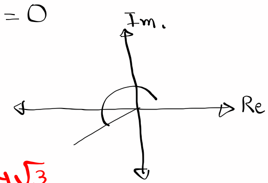


Jan 31-11:20 AM

Solve $x^3 - (-4 - 4i\sqrt{3}) = 0$

3 Answers

$x^3 = -4 - 4i\sqrt{3}$



$x = -4$

$y = -4\sqrt{3}$

$r = \sqrt{(-4)^2 + (-4\sqrt{3})^2} = \sqrt{16 + 16 \cdot 3} = \sqrt{64} = 8$

$\theta = 180^\circ + \tan^{-1}\left(\frac{-4\sqrt{3}}{-4}\right)$

$= 180^\circ + \tan^{-1}\sqrt{3} = 180^\circ + 60^\circ$

$= 240^\circ$

$x^3 = 8\,\text{Cis}\,240^\circ$

$x = \sqrt[3]{8\,\text{Cis}\,240^\circ}$

$x = 2\,\text{Cis}\,\alpha$

$\alpha = \frac{240^\circ + 360^\circ \cdot K}{3}$

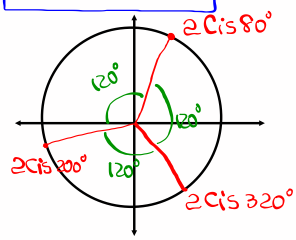
$\alpha = 80^\circ + 120^\circ K$

$K = 0 \rightarrow \alpha = 80^\circ$

$K = 1 \rightarrow \alpha = 200^\circ$

$K = 2 \rightarrow \alpha = 320^\circ$

$$\boxed{\begin{array}{l} 2\,\text{Cis}\,80^\circ \\ 2\,\text{Cis}\,200^\circ \\ 2\,\text{Cis}\,320^\circ \end{array}}$$



Jan 31-11:30 AM

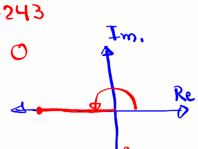
Solve $x^5 + 243 = 0$

5 Ans.

$-243 = -243 + 0i$

Re. -243

Im. 0



$x^5 = -243$

$x^5 = 243\,\text{Cis}\,180^\circ$

$r = 243$, $\theta = 180^\circ$

$x = \sqrt[5]{243\,\text{Cis}\,180^\circ}$

$\alpha = \frac{180^\circ + 360^\circ \cdot K}{5}$

$x = 3\,\text{Cis}\,\alpha$

$\alpha = 36^\circ + 72^\circ \cdot K$

$K = 0 \rightarrow \alpha = 36^\circ$

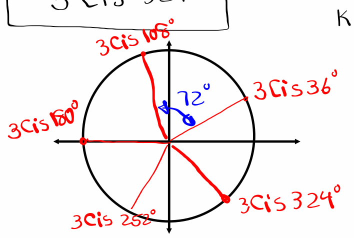
$K = 1 \rightarrow \alpha = 108^\circ$

$K = 2 \rightarrow \alpha = 180^\circ$

$K = 3 \rightarrow \alpha = 252^\circ$

$K = 4 \rightarrow \alpha = 324^\circ$

$$\boxed{\begin{array}{l} 3\,\text{Cis}\,36^\circ \\ 3\,\text{Cis}\,108^\circ \\ 3\,\text{Cis}\,180^\circ \\ 3\,\text{Cis}\,252^\circ \\ 3\,\text{Cis}\,324^\circ \end{array}}$$



Jan 31-11:40 AM